Theorem: The Form of a Convergent Power Series

If f is represented by a power series $f(x)=\sum a_n(x-c)^n$ for all x in an open interval I containing c, then $a_n=\frac{f^{(n)}(c)}{n!}$ and

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

Definitions of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x=c$, then the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

is called the **Taylor series for** $f(x)$ **at** c. Moreover, if $c=0$, then the series is the **Maclaurin series for** f.

1. Use the definition to find the Taylor series centered at 1 for the function $f(x)=e^x$.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n$$

$f'(x)=e^x$

$f''(x)=e^x$

$\vdots$

$f^{(n)}(x)=e^x$

$c=1,\quad f'(1)=e=f^{(n)}(1)$

$$f(x)=\sum_{n=0}^{\infty}\frac{e}{n!}(x-1)^n$$

$$\boxed{f(x)=e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!},\quad(-\infty,\infty)}$$

Ratio test

$$\lim_{n\to\infty}\left|\frac{(x-1)^{\cancel{n+1}\,1}}{\cancel{(n+1)!}_{\,n+1}}\cdot\frac{\cancel{n!}}{\cancel{(x-1)^n}}\right|=0<1\ \checkmark$$

Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all *x* in the interval *I*, then the Taylor series for *f* converges to

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. Prove that the Maclaurin series for $f(x) = \cosh x$ converges to the function for all *x*.

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Guidelines for Finding a Taylor Series

1. Differentiate $f(x)$ several times and evaluate each derivative at *c*. Try to recognize a pattern in these numbers.
2. Use the sequence developed in step 1 to form the Taylor coefficients $a_n = \frac{f^{(n)}(c)}{n!}$ and determine the interval of convergence for the resulting power series $f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$
3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Power Series for Elementary Functions

FUNCTION	INTERVAL OF CONVERGENCE
$\frac{1}{x}=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-\cdots+(-1)^n\cancel{(x-1)^n}\,x^n+\cdots$	$-1<x<1$
$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$	$0<x\le 2$
$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$
$\arcsin x=x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$

$(1+x)^k = 1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\frac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$	$-1<x<1$ **(depending on the value of *k*)**

$c=0$

$g(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \rightarrow$ Mac: $g(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

3. Find the Maclaurin series for the function $f(x)=x\cos x$.

1) Differentiate til you find a pattern and 2

$f(x)=x[\cos x]$

Let $g(x)=\cos x$

$g(0)=1$

$g'(x)=-\sin x,\ g'(0)=0$

$g''(x)=-\cos x,\ g''(0)=-1$

$g'''(x)=\sin x,\ g'''(0)=0$

$g(x)=\frac{1}{0!}x^0+\frac{0}{1!}x^1+\frac{-1}{2!}x^2+\frac{0}{3!}x^3+\cdots+-$

$g(x)=\frac{x^0}{0!}-\frac{x^2}{2!}+\frac{x^4}{4!}+-\cdots$

$g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

pattern is $1,0,-1,0$

So $f(x)=xg(x)=x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^1\cdot x^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$

3) Test

Ratio test: $\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{[2(n+1)]!}\cdot\frac{(2n)!}{x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{(2n)!}}{x^{2n+1}}\right|$

$=\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right|=0<1$ ✓

$\boxed{\text{So, } f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!},\ (-\infty,\infty)}$